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RESEARCH REPORT No. EM-130

Diffraction by a Spheroid

BERTRAM R. LEVY and JOSEPH B. KELLER

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DIFFRACTION BY A SPHEROID

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Abstract

The diffraction of a spherical scalar wave by a hard or soft spheroid is investigated theoretically. First the diffracted field is determined by the geometrical theory of diffraction[1]. Then for comparison the corresponding boundary value problem is solved exactly in terms of a series of products of spheroidal functions. The series involves the "radial" eigenfunctions which correspond to appropriate complex eigenvalues. Asymptotic expansions are derived for these functions for large values of the variable and the parameter. When used in the series solution, these expansions yield the asymptotic form of the diffracted field for incident wavelengths small compared to the spheroid dimensions. This result coincides precisely with that given by the geometrical theory. This agreement provides another verification of that theory. The expression for the field is used to calculate the back scattering and the field on the spheroid. The electromagnetic back scattering is finally computed with the aid of a theorem which relates it to the two scalar results. Errors in a previous treatment of this problem are also pointed out.

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1. Introduction

A spheroid is a solid of revolution generated by rotating an ellipse about one of its axes. We consider the problem of determining the field which results when a wave is incident upon a spheroid. The customary manner of dealing with such a problem involves formulating it as a boundary value problem for the reduced wave equation. This problem is then solved by means of a series of products of spheroidal wave functions. However this well known solution is practically useless when the wavelength of the incident field is small compared to the dimensions of the spheroid. It is just this case which we treat here, and therefore we must use a different procedure. In fact, we use two different procedures, both of which yield the desired result.

Our first result, given in Section 2, is based upon the geometrical theory of diffraction[1]. This is a theory which describes the field scattered by any object directly in terms of geometrical properties of the object and certain coefficients determined from a few canonical problems. We have previously applied this theory to diffraction by convex cylinders[2] and convex solids[3]. Whenever possible we compared the results of that theory with the asymptotic form (for wavelength small compared to object dimensions) of the solution of the corresponding boundary value problem. In every case the agreement was perfect, including that of the elliptic cylinder[4]. Although it has not yet been proved that this agreement will always occur, we are convinced that this is the case.

As a consequence of our work, we were surprised to read in the report of R. K. Ritt[5] that his asymptotic evaluation of the diffracted field on the surface of a spheroid disagreed with our geometric theory. Feeling certain that his work must contain an error, we decided to rederive the asymptotic form of this field. Our derivation is given in Section 3. The derivation employs certain new asymptotic formulas for spheroidal functions which are deduced in Appendix I. The corresponding formulas used by Ritt, which were derived by N. D. Kazarinoff[5], are incorrect. In Appendix III we point out the errors in their derivation, as well as some other mistakes in the use of them. Our derivation leads to an asymptotic form of the field which coincides precisely with the result of the geometric theory given in Section 2. However, to recognize the identity of the two results, it is necessary to utilize certain addition formulas for elliptic integrals.

As applications of our result, we compute the field scattered back to the source, and then find the back scattering cross-section. We also evaluate the field on the surface of the spheroid.

All of our results pertain to a prolate spheroid. The incident field is produced by a point source on the axis of revolution. The field is a scalar which satisfies the reduced wave equation and the spheroid is either hard or soft. In the case of back-scattering, the cross-section for an electromagnetic wave hitting a perfectly conducting spheroid is also given. This calculation is based upon a theorem proved in Appendix II. It relates the electromagnetic back scattering for axial incidence on a body of revolution, to the corresponding scalar results for hard and soft bodies.

2. Geometric construction of the diffracted field

Let us consider a prolate spheroid in a homogeneous medium, having propagation speed c. Suppose that a point source of radiation, located on the axis of revolution of the spheroid, emits a wave of angular frequency ω. Then $k = \omega/c$ is the propagation constant of the medium for such a wave. When the incident spherical wave strikes the spheroid, reflected and diffracted fields are produced. The reflected field u_r can be constructed by the method of geometrical optics, as described, for example, in reference [6]. The diffracted field u_d can be constructed by means of the geometrical theory of diffraction, as has been done in reference [3] for diffraction by any smooth convex scattering object. When that result is specialized to the case of a point source on the axis of a body of revolution it becomes

$$(1) \qquad u_d(P) = \frac{e^{ikR_o}}{4\pi R_o}\left[\left\{\frac{r_o\rho_1}{s_1r_1(\rho_1-s_1)}\right\}^{1/2}\sum_m \frac{D_m(P_1)D_m(Q_1)}{1+\exp\left[ikT-\int_o^T \alpha_m d\tau\right]}\exp\left[ik(t_1+s_1)-\int_{Q_1}^{P_1}\alpha_m d\tau\right] - i\left\{\frac{r_o\rho_2}{s_2r_2(s_2-\rho_2)}\right\}^{1/2}\sum_m \frac{D_m(P_2)D_m(Q_2)}{1+\exp\left[ikT-\int_o^T \alpha_m d\tau\right]}\exp\left[ik(t_2+s_2)-\int_{Q_2}^{P_2}\alpha_m d\tau\right]\right]$$

In (1) P denotes the point at which the field is evaluated. The plane containing P and the axis we will call the xy plane. Then s_1 and s_2 denote the lengths of the tangents from P to the spheroid in this plane and P_1, P_2 denote the corresponding points of tangency. (See Figure 1) The points Q_1 and Q_2 are the points of tangency of the tangents from the source point Q to the spheroid in this plane. The distances from the axis to P_1, P_2 and Q_1 or Q_2 are denoted by r_1, r_2 and r_o respectively. The total arclength of the generating ellipse is denoted by T, while t_1 is the arclength along the ellipse from Q_1 to P_1 and t_2 is the arclength from Q_2 to P_2. The length of the tangent from P_1 to the point

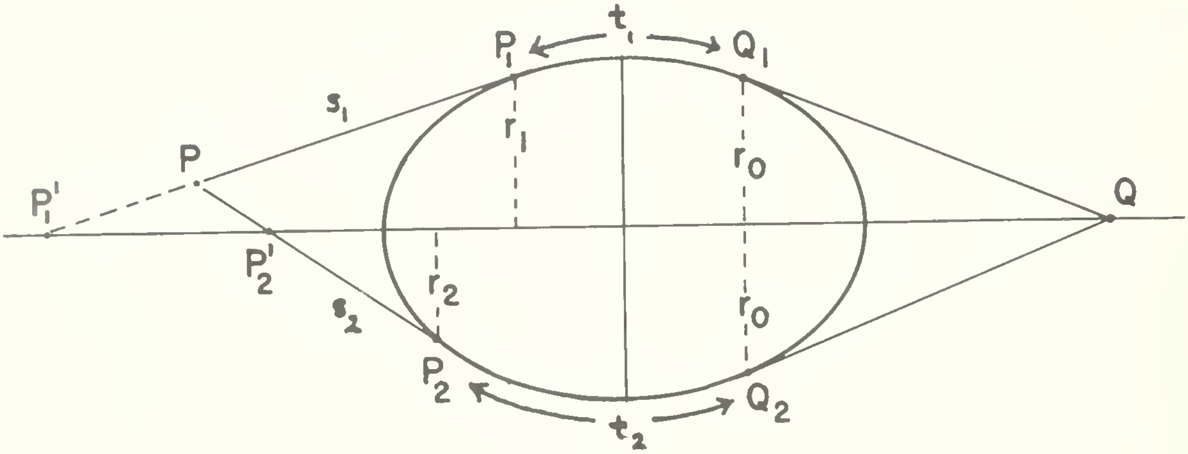

Figure 1

A cross section of the ellipsoid showing the source Q and a field point P. Two rays from Q are tangent to the ellipsoid at Q_1 and Q_2 where they produce diffracted surface rays. The ray from Q_1 travels a distance t_1 along the ellipsoid to the point P_1 from which it sheds a diffracted ray which passes through P. In addition it continues to encircle the ellipsoid and sheds another ray through P each time it passes P_1. The ray from Q_2 sheds rays through P each time it passes through P_2. These rays pass through the axial caustic at P_2' before reaching P.

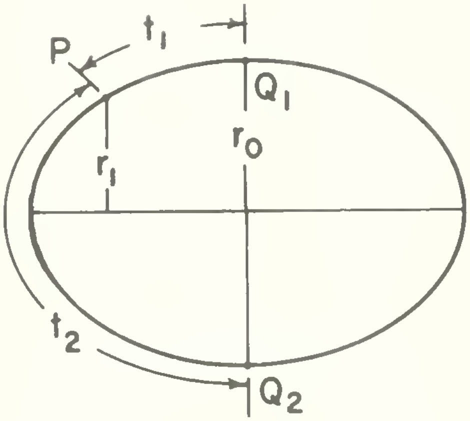

Figure 2

A cross section of the ellipsoid showing the points Q_1 and Q_2 at which two incident rays are tangent to it. The incident field is a plane wave coming from the right. The tangent rays produce diffracted rays which travel distances t_1 and t_2 to a point P on the surface.

P_1' at which it intersects the axis is denoted by ρ_1, while ρ_2 denotes the distance from P_2 to P_2'. The integrals are with respect to arclength τ along the generating ellipse. The decay coefficient $\alpha_m(\tau)$ is defined in terms of $b(\tau)$, the radius of curvature of the ellipse, by

$$(2) \quad \alpha_m = e^{-i\pi/6} (k/6)^{1/3} q_m b^{-2/3}$$

The constant q_m is defined in Appendix 1. For the soft spheroid, $q_m \equiv q_m^{(1)}$ while for the hard spheroid $q_m \equiv q_m^{(2)}$. When it becomes necessary to distinguish between these two cases the coefficient α_m must be given the same superscript as q_m. The diffraction coefficient D_m depends upon the radius of curvature at the point at which D_m is evaluated. For the soft spheroid $D_m \equiv D_m^{(1)}$ is given by

$$(3) \quad D_m^{(1)} = \pi^{3/4} 2^{1/4} 6^{-2/3} k^{-1/12} \left[A'(q_m^{(1)})\right]^{-1} e^{i\pi/24} b^{1/6}$$

For the hard spheroid $D_m \equiv D_m^{(2)}$ is given by

$$(4) \quad D_m^{(2)} = \pi^{3/4} 2^{-1/4} 6^{-1/6} k^{-1/12} \left[q_m^{(2)1/2} A(q_m^{(2)})\right]^{-1} e^{i\pi/24} b^{1/6}$$

The Airy function A is defined in Appendix 1.

The result (1) can be rewritten in another form by introducing prolate spheroidal coordinates ξ, η, ϕ. These coordinates are most easily defined by relating them to the Cartesian coordinates x,y,z. Let the axis of symmetry of the spheroid be the x axis and let the origin be at the center of the spheroid. Finally, let 2h denote the interfocal distance of the generating ellipse. Then the prolate spheroidal coordinates may be defined by

$$x = h \cosh \xi \cos \eta$$
$$y = h \sinh \xi \sin \eta \cos \phi$$
$$z = h \sinh \xi \sin \eta \sin \phi$$

The angular variables η and ϕ range from 0 to π and 0 to 2π respectively, while $0 \leq \xi < \infty$. The spheroid may be defined by the equation $\xi = a$. The point source Q will be assumed to lie at $\eta = 0$, $\xi = \xi_o \geq a$. The η coordinates of the points of tangency Q_1, P_1, Q_2 and P_2 will be denoted by η_1, η_2, η_3 and η_4 respectively. Their other coordinates are $\xi = a$ and either $\phi = 0$ or $\phi = \pi$.

The geometrical quantities in (1) can be expressed in terms of these coordinates. Most of them have previously been calculated in reference [4] and the remaining ones can be easily found. When these expressions are inserted into (1), the result for the field diffracted by the soft spheroid becomes

$$(5) \quad u_d(P) = \pi^{1/2}\, 2^{-3/2}\, 6^{-4/3}\, e^{i\pi/12}\, k^{-1/6}\, h^{-7/6}\, (\cosh a \sinh a)^{1/6}$$
$$(\sinh \xi \sinh \xi_o \sin \eta)^{-1/2}\, (\cosh^2 \xi_o - \cosh^2 a)^{-1/4}$$
$$\sum_{n=o}^{\infty} \left[f_n(a, \eta_1, \eta_2) - i f_n(a, -\eta_3^*, -\eta_4^*)\right] \left[A'(q_n^{(1)})\right]^{-2}$$
$$\left\{1 + \exp\left[ikh \int_0^{2\pi} (\cosh^2 a - \cos^2 \eta)^{1/2}\, d\eta + i\tau_n^{(1)} (kh)^{1/3} (\cosh a \sinh a)^{2/3} \right.\right.$$
$$\left.\left. \int_0^{2\pi} (\cosh^2 a - \cos^2 \eta)^{-1/2} d\eta\right]\right\}^{-1}.$$

In (5), $\tau_n^{(1)} = 6^{-1/3} e^{i\pi/3} q_n^{(1)}$ and $\eta_3^* = 2\pi - \eta_3$, while $\eta_4^* = \eta_4$ if the $\emptyset$ coordinate of P_2 is 0 and $\eta_4^* = 2\pi - \eta_4$ if the $\emptyset$ coordinate of P_2 is π. We assume that the $\emptyset$ coordinate of P is 0. The function f_n is given by

$$(6) \quad f_n = (\cosh^2 a - \cos^2\eta_2)^{1/4} \Big[(\cosh\xi \cos\eta - \cosh a \cos\eta_2)^2 + (\sinh\xi \sin\eta - \sinh a \sin\eta_2)^2\Big]^{-1/4} \exp\Big[ikh \tanh\xi_o(\cosh^2\xi_o - \cosh^2 a)^{1/2} + ikh\big[(\cosh\xi\cos\eta - \cosh a\cos\eta_2)^2 + (\sinh\xi\sin\eta - \sinh a\sin\eta_2)^2\big]^{1/2} + ikh\int_{\eta_1}^{\eta_2}(\cosh^2 a - \cos^2\eta)^{1/2} d\eta + i\tau_n^{(1)}(kh)^{1/3}(\cosh a \sinh a)^{2/3} \int_{\eta_1}^{\eta_2}(\cosh^2 a - \cos^2\eta)^{-1/2} d\eta\Big]$$

For the hard spheroid (5) applies if we replace $\tau_n^{(1)}$ by $\tau_n^{(2)}$ and $\left[A'(q_n^{(1)})\right]^{-2}$ by $3\left[q_n^{(2)} A^2(q_n^{(2)})\right]^{-1}$.

The preceding result (1) or (5) is not valid on the surface of the spheroid nor on the axis of symmetry because both of these places are caustics of the diffracted rays. However, it can easily be modified to be valid in these places by applying the appropriate caustic corrections. [3] To obtain the field on the surface, we must multiply the m^th term in the first sum in (1) by the factor

$$(7) \qquad R_m = 2^{1/2} 6^{1/3} \pi^{-1/2} k^{1/6} s_1^{1/2} A(q_m^{(2)}) \exp\left[-iks_1 - i\pi/12\right] b^{1/3}(P_1) \quad ,$$

multiply the m^th term of the second sum in (1) by R_m with s_1 replaced by s_2 and P_1 replaced by P_2, and then let s_1 and s_2 -> 0. Upon doing this and noting that P_1, P_2 and P become identical, while $r_1 = r_2$ and $P_1 = P_2$, we find that (1) becomes on the surface of the hard spheroid

$$(8)\quad u_d(P) = e^{ikR_o}(4\pi R_o)^{-1}2^{1/2}6^{1/3}\pi^{-1/2}k^{1/6}e^{-i\pi/12}(r_o/r_1)^{1/2}b^{-1/3}(P)$$

$$\sum_m \frac{D_m(P)A(q_m)}{1+\exp\left[ikT-\int_0^T \alpha_m d\tau\right]}\left\{D_m(Q_1)\exp\left[ikt_1-\int_{Q_1}^P \alpha_m d\tau\right]\right.$$

$$\left.-iD_m(Q_2)\exp\left[ikt_2-\int_{Q_2}^P \alpha_m d\tau\right]\right\}$$

For later use it is convenient to specialize (8) to the case of plane wave incidence. If the incident plane wave is given by $u_i = e^{-ikx}$ then the diffracted field is given by (8) with $e^{ikR_o}(4\pi R_o)^{-1}$ replaced by 1. The points Q_1 and Q_2 become the ends of the minor axis of the generating ellipse of the spheroid. Thus, when we write (8) in terms of spheroidal coordinates for plane wave incidence, we obtain

$$(9)\quad u_d(P) = \frac{\pi(\cosh a)^{1/2}}{(\sin\eta)^{1/2}(\cosh^2 a-\cos^2\eta)^{1/4}}\sum_{m=1}^{\infty}\frac{\exp\left[G(\pi/2,\eta)\right]-i\exp\left[G(\eta,3\pi/2)\right]}{\left(1+\exp\left[G(0,2\pi)\right]\right)q_m^{(2)}A(q_m^{(2)})}\ ,$$

Here we have introduced the function $G(\alpha,\beta)$ which is defined by

$$(10)\quad G(\alpha,\beta) = ikh\int_\alpha^\beta(\cosh^2 a-\cos^2\eta)^{1/2}d\eta + i(kh)^{1/3}\tau_n^{(2)}(\sinh a\cosh a)^{2/3}$$

$$\int_\alpha^\beta(\cosh^2 a - \cos^2\eta)^{-1/2}d\eta$$

To calculate the back scattering cross section of the spheroid, we again specialize (5) to plane wave incidence with $u_{inc} = e^{-ikx}$. We then apply the method of reference [3] to correct u_d on the axial caustic. When this is done we find that the leading term in the expansion of $u_d(P)$ for a point on the axis becomes, for large values of x,

$$(11)\quad \begin{gathered} u_d(P) = -\delta e^{ikx + i\pi/3} \exp\left[i\tau (kh)^{1/3} (\cosh a \sinh a)^{2/3} \int_{\pi/2}^{3\pi/2} (\cosh^2 a - \cos^2 \eta)^{-1/2} d\eta \right] \\ \pi^2 k^{1/3} h (\sinh a \cosh a)^{2/3} \left\{ xq A^2 6^{1/3} \right\}^{-1} . \end{gathered}$$

In (11), $\tau = 6^{-1/3} e^{i\pi/3} q$. The values of δ, q and A or A' are given below for the two cases.

$$(12)\quad \left.\begin{aligned} \delta &= 1 \\ q &= 1.469354 \\ A &= 1.16680 \end{aligned}\right\} \quad \text{for the hard spheroid}$$

$$(13)\quad \left.\begin{aligned} \delta &= qA^2 3^{-1} (A')^{-2} \\ q &= 3.372134 \\ A' &= -1.059053 \end{aligned}\right\} \quad \text{for the soft spheroid}$$

The total back scattered field u is the sum of the diffracted field $u_d(P)$ and the geometrically reflected field $u_g(P)$. To calculate $u_g(P)$ we use the methods of reference [6]. We then obtain for the leading term in $u_g(P)$ the result

$$(14)\quad u_g(P) = \pm \frac{h}{2} \frac{\cosh^2 a}{\sinh a} \exp\left[ik(x - 2h \cosh a) \right] .$$

Here the positive sign holds for the hard spheroid and the negative sign for the soft spheroid. Upon combining (11) and (14), we obtain

$$(15)\quad u(P) = \pm \frac{h}{2}\frac{\cosh^2 a}{\sinh a}\exp\left[ik(x-2h\cosh a)\right]$$
$$\left\{1 - \frac{2\delta\pi^2(kh)^{1/3}(\sinh a)^{5/3}}{qA^2 6^{1/3}(\cosh a)^{4/3}}\exp\left[2ikh\cosh a + i\pi/3\right.\right.$$
$$\left.\left. + i\tau(kh)^{1/3}(\cosh a\sinh a)^{2/3}\int_{\pi/2}^{3\pi/2}(\cosh^2 a - \cos^2\eta)^{-1/2}d\eta\right]\right\} .$$

Here the plus sign and (12) hold for the hard spheroid and the negative sign and (13) hold for the soft spheroid.

3. Analytic Solution

In terms of our previously defined spheroidal coordinates the reduced wave equation corresponding to a point source located at $\xi = \xi_0$, $\eta = 0$ is

$$(1)\quad u_{\xi\xi} + u_{\eta\eta} + u_\xi \coth\xi + u_\eta \cot\eta + (kh)^2(\cosh^2\xi - \cos^2\eta)u = \frac{\delta(\eta)\delta(\xi-\xi_0)}{2\pi h\sinh\xi_0 \sin\eta}$$

In (1) we have made use of the fact that u is independent of ϕ. The boundary value problem to be solved is to find a solution $u(\xi,\eta)$ of (1) which satisfies the Sommerfeld radiation condition at $\xi = \infty$. On the surface of the spheroid u must satisfy either of the following boundary conditions

$$(2)\quad u(a,\eta) = 0 \qquad \text{(acoustically soft spheroid)}$$

or

$$(3)\quad u_\xi(a,\eta) = 0 \qquad \text{(acoustically hard spheroid)}$$

If we replace the right hand side of (1) by zero, the resulting equation has product solutions of the form $\Phi(\xi)\Psi(\eta)$. The functions Φ and Ψ satisfy the equations

$$(4)\quad \frac{d^2\Phi}{d\xi^2} + \coth\xi\,\frac{d\Phi}{d\xi} - (kh)^2(b^2 - \cosh^2\xi)\,\Phi = 0 \quad ,$$

$$(5)\quad \frac{d^2\Psi}{d\eta^2} + \cot\eta\,\frac{d\Psi}{d\eta} + (kh)^2\,(b^2 - \cos^2\eta)\,\Psi = 0 \quad .$$

In (4) and (5) b is an arbitrary separation constant. We now note that for certain complex values b_n of b, (4) has outgoing solutions $V_n^{(1)}(\xi)$, which satisfy either of the conditions $V_n^{(1)}(a) = 0$ or $V_n^{(1)\prime}(a) = 0$.* Consequently, we may try to solve the boundary value problem by assuming that u has the form

$$(6)\quad u(\xi,\eta) = \sum_{n=1}^{\infty} W_n(\eta)\, V_n^{(1)}(\xi).$$

If we substitute (6) into (1) and make use of the orthogonality of the $V_n^{(1)}$, we find that W_n satisfies the equation

$$(7)\quad \frac{d^2W_n}{d\eta^2} + \cot\eta\,\frac{dW_n}{d\eta} + (kh)^2(b_n^{\,2} - \cos^2\eta)W_n = C_n\,\frac{\delta(\eta)}{\sin\eta} \quad .$$

In (7) the constant C_n has the value

$$(8)\quad C_n = V_n^{(1)}(\xi_o)\left(2\pi h\int_a^{\infty}\left[V_n^{(1)}(\xi)\right]^2 \sinh\xi\, d\xi\right)^{-1} \quad .$$

The orthogonality condition used in obtaining (7) is

$$(9)\quad \int_a^{\infty} V_n^{(1)}(\xi)V_m^{(1)}(\xi)\sinh\xi\,d\xi = \delta_{nm}\int_a^{\infty}\left[V_n^{(1)}(\xi)\right]^2 \sinh\xi\,d\xi \quad .$$

---

*When it becomes necessary to distinguish between the sets of eigenvalues we will use $b_n^{(1)}$ to denote eigenvalues corresponding to the boundary condition (2) and $b_n^{(2)}$ to denote eigenvalues corresponding to the boundary condition (3).

This condition follows from (4) and the boundary conditions satisfied by the $V_n^{(1)}$. An evaluation similar to the one carried out in reference [4] shows that for the soft spheroid

$$(10)\quad \int_a^\infty [V_n^{(1)}(\xi)]^2 \sinh\xi \, d\xi = [2(kh)^2 b_n^{(1)}]^{-1} V_n^{(1)\prime}(a) \sinh a \frac{\partial}{\partial b} V_n^{(1)}(a) \quad .$$

For the hard spheroid we obtain instead

$$(11)\quad \int_a^\infty [V_n^{(1)}(\xi)]^2 \sinh\xi \, d\xi = -[2(kh)^2 b_n^{(2)}]^{-1} V_n^{(1)}(a) \sinh a \frac{\partial}{\partial b} V_n^{(1)\prime}(a) \quad .$$

In order to solve (7) we let $W_n^{(1)}(\eta)$ be a solution of the homogeneous equation obtained from (7) by setting $C_n = 0$. We require $W_n^{(1)}(\eta)$ to be regular at $\eta = 0$. A simple calculation then shows that $W_n^{(1)}(\eta)$ is an even function of η and that $W_n^{(1)}(\pi - \eta)$ is also a solution of the homogeneous equation. Since $\eta = 0$ is a singular point of (7) the subsequent analysis will be simplified by shifting the delta function to $\eta = \eta_o$ and, finally, letting $\eta_o \to 0$. Then the conditions determining $W_n(\eta)$ are that it be regular at $\eta = 0$ and $\eta = \pi$, that it be continuous across $\eta = \eta_o$ and that $W_n'(\eta)$ have a jump of magnitude $C_n(\sin\eta_o)^{-1}$ across $\eta = \eta_o$. By applying these conditions we find that $W_n(\eta)$ is given by

$$(12)\quad \begin{aligned} W_n(\eta) &= -C_n[\sin\eta_o \, \omega_n(\eta_o)]^{-1} W_n^{(1)}(\eta_o) \, W_n^{(1)}(\pi - \eta) \quad , & \eta_o \le \eta \le \pi \\ W_n(\eta) &= -C_n[\sin\eta_o \, \omega_n(\eta_o)]^{-1} W_n^{(1)}(\pi - \eta_o) \, W_n^{(1)}(\eta) & 0 \le \eta \le \eta_o \quad . \end{aligned}$$

Here $\omega_n(\eta)$ is defined by

$$(13)\quad \omega_n(\eta) = W_n^{(1)}(\pi - \eta) W_n^{(1)\prime}(\eta) + W_n^{(1)}(\eta) \, W_n^{(1)\prime}(\pi - \eta) \quad .$$

We note that ω_n is the Wronskian of two linearly independent solutions of the homogeneous equation. Therefore, a simple calculation shows that it has the value

$$(14) \qquad \omega_n(\eta) = \frac{\omega_n(\pi/2)}{\sin \eta} = 2\, W_n^{(1)}(\pi/2)\, W_n^{(1)\prime}(\pi/2)$$

Upon using (14) in (12) and letting $\eta_o \to 0$ we obtain

$$(15) \qquad W_n(\eta) = -C_n \left[2W_n^{(1)}(\pi/2)\, W_n^{(1)\prime}(\pi/2) \right]^{-1} W_n^{(1)}(0)\, W_n^{(1)}(\pi - \eta)$$

We can obtain another expression for $W_n(\eta)$ by using (13) directly in (12). Upon doing this and using the fact that $W_n^{(1)}(\eta)$ has a logarithmic singularity at $\eta = \pi$ *, we obtain

$$(16) \qquad W_n(\eta) = -C_n \left[\lim_{\eta_o \to 0} \sin \eta_o\, W_n^{(1)\prime}(\pi - \eta_o) \right]^{-1} W_n^{(1)}(\pi - \eta) \quad .$$

In Appendix I, equation (18), we have evaluated the limit appearing in (16). We use this result, together with (16), (10) and (8) in (6). Then, in the case of the soft spheroid, (6) becomes

$$(17) \qquad u(\xi,\eta) = -\frac{k^2 h}{2 \sinh a} \sum_{n=1}^{\infty} \frac{b_n V_n^{(1)}(\xi_o) V_n^{(1)}(\xi)\, W_n^{(1)}(\pi - \eta)}{\cos\left[k \int_0^{\pi} (b_n^2 - \cos^2\eta)^{1/2} d\eta \right] V_n^{(1)\prime}(a)\, \frac{\partial}{\partial b} V_n^{(1)}(a)}$$

For the case of the hard spheroid (17) still applies if the right side is

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*See Appendix I

multiplied by -1 and if $V_n^{(1)}(a)$ and $V_n^{(1)'}(a)$ are interchanged.

We shall now obtain the asymptotic form of u for large values of kh by using the asymptotic expansions of the spheroidal functions given in Appendix I. We find, in the case of the soft spheroid, that (17) has the asymptotic form

$$(18)\quad u(\xi,\eta) \sim -\pi^{1/2} 2^{-3/2} 6^{-4/3} e^{i\pi/12} k^{-1/6} h^{-7/6} (\sinh\xi \sinh\xi_o \sin\eta)^{-1/2}$$

$$(\cosh a \sinh a)^{2/3} (\cosh^2\xi_o - \cosh^2 a)^{-1/4} (\cosh^2\xi - \cosh^2 a)^{-1/4} (\cosh^2 a - \cos^2\eta)^{-1/4}$$

$$\sum_{n=1}^{\infty} \left[A'(q_n^{(1)})\right]^{-2} \exp\left[iF(\xi) + iF(\xi_o)\right] \frac{\exp[iG(\eta)] - i\exp[-iG(\eta) + 2iG(\pi)]}{1 + \exp[2iG(\pi)]} .$$

In (18) we have introduced $F(\xi)$ and $G(\eta)$ which are defined by

$$(19)\quad F(\xi) = kh \int_a^{\xi} (\cosh^2\xi - \cosh^2 a)^{1/2} - i(kh)^{1/3} \tau_n^{(1)} (\sinh a \cosh a)^{2/3}$$

$$\int_a^{\xi} (\cosh^2\xi - \cosh^2 a)^{-1/2} d\xi$$

$$(20)\quad G(\eta) = kh \int_o^{\eta} (\cosh^2 a - \cos^2\eta)^{1/2} d\eta + i(kh)^{1/3} \tau_n^{(1)} (\sinh a \cosh a)^{2/3}$$

$$\int_o^{\eta} (\cosh^2 a - \cos^2\eta)^{-1/2} d\eta .$$

In the case of the hard spheroid (18) - (20) apply if $\tau_n^{(1)}$ is replaced by $\tau_n^{(2)}$ and $[A'(q_n^{(1)})]^{-2}$ is replaced by $3\left[q_n^{(2)} A^2(q_n^{(2)})\right]^{-1}$.

It is a simple matter to show that the result for u given in (5) of section 2, obtained by the geometric theory of diffraction, agrees precisely with (18) obtained by asymptotically expanding the exact solution.

The necessary calculations are contained in reference [4]. The identity of the amplitudes follow from equation (I-1) of reference [4], while the identity of the phases follows from (54), (55), (56) and (57) of that paper.

Let us now calculate the diffracted field at a point $P(a,\eta)$ on the surface of a hard spheroid due to an incident plane wave e^{-ikx}. To do this, we write (17) for a hard spheroid, set $\xi = a$, multiply by $4\pi h \cosh \xi_o \exp\left[-ikh \cosh \xi_o\right]$ and let $\xi_o \to \infty$ to obtain

$$(21)\quad u(a,\eta) \sim \frac{k^{11/6} h^{11/6} \pi^{1/2} e^{i\pi/12}}{\sinh a} \sum_{n=1}^{\infty} \frac{b_n W_n^{(1)}(\pi-\eta) \exp\left[ikh\, f(b_n)\right]}{\cos\left[k \int_o^{\pi} (b_n^2 - \cos^2\eta)^{1/2} d\eta\right] \frac{\partial}{\partial b} (V_n^{(1)\prime}(a))} .$$

Here

$$(22)\quad f(b_n) = \lim_{\xi_o \to \infty} \left(\int_{\cosh^{-1} b_n}^{\xi_o} (\cosh^2 \xi - b_n^2)^{1/2} d\xi - \cosh \xi_o \right) .$$

Using the methods of Appendix I we find

$$(23)\quad \frac{\partial}{\partial b} V_n^{(1)\prime}(a) \sim -\pi^{-1} (kh)^{4/3} 2^{1/2} (\sinh a)^{-1} (\cosh a)^{1/2} e^{-2i\pi/3} q_n A(q_n) .$$

The evaluation of $f(b_n)$ has been carried out in a slightly different notation by Ritt [5] and yields the result

$$(24)\quad f(b_n) = -\int_o^{\pi/2} (b_n^2 - \cos^2\eta)^{1/2} d\eta .$$

Using (23) and (24) in (21), together with the asymptotic expansion of

$W_n^{(1)}$ contained in Appendix I, we obtain

$$(25) \qquad u(a,\eta) \sim \frac{(\cosh a)^{1/2}\pi}{(\sin\eta)^{1/2}(\cosh^2 a - \cos^2\eta)^{1/4}} \sum_{n=1}^{\infty} \frac{e^{iG(\eta)} - ie^{2iG(\pi) - iG(\eta)}}{\left(1 + e^{2iG(\pi)}\right) q_n^{(2)} A(q_n^{(2)})} e^{-iG(\pi/2)}$$

Here $G(\eta)$ is defined by (20). This result (25) is identical with equation (9) of section 2, obtained by the geometric theory.

4. Conclusion and electromagnetic back scattering.

We have determined the scalar field diffracted by a hard or soft spheroid of dimensions large compared to the incident wavelength. This field is given by (1) or (5) at points off the spheroid surface and off the axis. On the spheroid surface it is given by (8) for the hard spheroid and by (9) if the incident field is a plane wave. The back scattered field on the axis is given by (12) for an incident plane wave. All these results have been derived by using the geometric theory of diffraction and, independently, by asymptotically expending the solution of the appropriate boundary value problem. They are in a form which is suitable for numerical evaluation.

As an application of our results we shall now calculate the back scattered field from a perfectly conducting spheroid when a plane electromagnetic wave is incident along the axis. To do this we could apply the geometric theory of diffraction directly. However, it is simpler to utilize our previous results by means of the theorem proved in Appendix II. According to that theorem

$$(1) \qquad E = \tfrac{1}{2}(u_S - u_H)\, E_i \;.$$

Here E is the back scattered electromagnetic field due to the incident field $E_i e^{-ikx}$. The scalar back scattered fields are u_S and u_H for the soft and hard spheroids respectively, with the field e^{-ikx} incident. Upon using (15) of Section 2 for u_S and u_H, (1) yields

$$
\begin{aligned}
(2) \quad E = \frac{-E_i}{2}\,\frac{h\cosh^2 a}{\sinh a}\exp\left[ik(x-2h\cosh a)\right] \Bigg\{ 1 - \frac{\pi^2 (kh)^{1/3}(\sinh a)^{5/3}}{6^{1/3}(\cosh a)^{4/3}} \\
\times \exp\left[2ikh\cosh a + i\pi/3\right] \Bigg(3^{-1} A'^{-2} \exp\Bigg[i\tau_1 (kh)^{1/3}(\cosh a \sinh a)^{2/3} \\
\times \int_{\pi/2}^{3\pi/2} (\cosh^2 a - \cos^2\eta)^{-1/2}\, d\eta \Bigg] + q_2^{-1} A^{-2} \exp\Bigg[i\tau_2 (kh)^{1/3} \\
\times (\cosh a \sinh a)^{2/3} \int_{\pi/2}^{3\pi/2} (\cosh^2 a - \cos^2\eta)^{-1/2}\, d\eta \Bigg] \Bigg) \Bigg\} .
\end{aligned}
$$

Here A and q_2 are given by (12) of Section 2, while A' and q_1 are given by (13) of that section. The last term in the brace can usually be neglected. From (2), the electromagnetic back scattering cross section can be computed.

Appendix I. Asymptotic expansions of the spheroidal functions

In this appendix we study the asymptotic expansions as kh → ∞ of the various solutions of equations (4) and (5) in Section 3. It will be convenient to denote kh by k. To analyze equation (4) of Section 3, we set

$$(1) \qquad \Phi(\xi) = (\sinh \xi)^{-1/2}\, \Gamma(\xi) \quad .$$

Then we find that Γ satisfies the equation

$$(2) \qquad \Gamma'' - k^2(b^2 - \cosh^2\xi + 2^{-1}k^{-2} - 4^{-1}k^{-2} \coth^2\xi)\, \Gamma = 0 \quad .$$

Since $\xi > a > 0$, we may apply the methods of F.W.J. Olver[7] to obtain uniform asymptotic expansions of $\Gamma(\xi)$. The calculation is similar to the one carried out in reference [4]. It yields in the present case for the outgoing solution, $V^{(1)}(\xi)$, of equation (4) of Section 3.

$$(3) \qquad V^{(1)}(\xi) \sim \zeta^{1/4} 3^{1/3} \pi^{-1} (b^2 - \cosh^2\xi)^{-1/4} (\sinh \xi)^{-1/2} A\left(3^{1/3} e^{-i\pi/3} k^{2/3} \zeta\right) .$$

In (3) ζ is defined by

$$(4) \qquad \frac{2}{3}\zeta^{3/2} = \int_{\cosh^{-1} b}^{\xi} (b^2 - \cosh^2 x)^{1/2}\, dx \; .$$

The Airy function A(t) is defined by

$$(5) \qquad A(t) = \int_0^\infty \cos(z^3 - tz)\, dz \quad .$$

In the case when $\xi > \cosh^{-1} b$ we can expand the Airy functions in (3) to obtain

$$(6) \qquad V^{(1)}(\xi) \sim e^{i\pi/2} k^{-1/6} 2^{-1} \pi^{-1/2} (\cosh^2 \xi - b^2)^{-1/4} (\sinh \xi)^{-1/2}$$

$$\times \exp\left[ik \int_{\cosh^{-1} b}^{\xi} (\cosh^2 \xi - b^2)^{1/2} d\xi \right] \quad .$$

The calculation of the eigenvalues, b_n, is exactly the same as in reference [4] and yields

$$(7) \qquad b_n \sim \cosh a + \tau_n k^{-2/3} (\sinh a)^{2/3} (\cosh a)^{-1/3} \quad .$$

Here τ_n is defined by

$$(8) \qquad \tau_n = 6^{-1/3} e^{i\pi/3} q_n \quad .$$

In (7) and (8) b_n, τ_n and q_n are to have superscripts 1 for the soft spheroid and supercripts 2 for the hard spheroid where $q_n^{(1)}$ is defined by $A(q_n^{(1)}) = 0$ and $q_n^{(2)}$ is defined by $A'(q_n^{(2)}) = 0$.

To obtain the asymptotic expansion of $W^{(1)}(\eta)$ we must proceed in a manner different from that which we followed in the case of the corresponding elliptic cylinder function. This is due to the singularities of equation (5) of Section 3 at $\eta = 0$ and $\eta = \pi$. Accordingly, we apply a boundary layer type of analysis to obtain the asymptotic expansion. However, we do not find it necessary to introduce a stretched independent variable because we consider only the leading term in the expansion. The use of such a variable would permit us to obtain further terms. In the neighborhood of $\eta = 0$, equation (5) of Section 3 becomes

$$(9) \qquad \psi'' + \frac{1}{\eta}\psi' + p^2\psi = 0 \quad ,$$

where

$$(10) \qquad p^2 = k^2(b^2 - 1) \quad .$$

Thus the solution regular at $\eta = 0$ is proportional to $J_o(p\eta)$. We may normalize it to have the value 1 at $\eta = 0$. Then

$$(11) \qquad \psi \sim J_o(p\eta) \quad \text{for} \quad \eta \ll 1 \quad .$$

Similarly, in the neighborhood of $\eta = \pi$, equation (5) of Section 3 has the form (9) with η replaced by $\pi - \eta$. Hence, we may conclude that

$$(12) \qquad \psi \sim A\, J_o(p(\pi-\eta)) + B\, N_o(p(\pi-\eta)) \quad \text{for} \quad \pi - \eta \ll 1 \quad .$$

In (12) A and B are as yet undetermined. Away from $\eta = 0$ and $\eta = \pi$, we may apply standard W.K.B. arguments to obtain

$$(13) \qquad \psi \sim \frac{C \cos\left[k \int_0^\eta (b^2 - \cos^2\eta)^{1/2}\, d\eta\right] + D \sin\left[k \int_0^\eta (b^2 - \cos^2\eta)^{1/2}\, d\eta\right]}{(\sin\eta)^{1/2}(b^2 - \cos^2\eta)^{1/4}} \quad .$$

The constants A,B,C and D are obtained by matching (11) and (12) to (13) in their respective common regions of validity. To obtain C and D we set $\eta = \epsilon k^{-1/2}$. Then the argument of J_o in (11) is large as $k \to \infty$ and we may expand J_o asymptotically for large argument. Upon doing this and requiring that the resulting expression be identical with (13) we obtain

$$(14) \qquad C = D = (\pi k)^{-1/2} \quad .$$

To obtain A and B we set $\eta = \pi - \epsilon k^{-1/2}$. Then upon expanding (12) asymptotically and comparing with (13) we obtain

$$(15) \qquad A = \sin\left[k \int_0^\pi (b^2 - \cos^2\eta)^{1/2} d\eta\right] \quad ,$$

and

$$(16) \qquad B = -\cos\left[k \int_0^\pi (b^2 - \cos^2\eta)^{1/2} d\eta\right] \quad .$$

Using (14) in (13) we obtain the following result, which is valid except in the neighborhoods of $\eta = 0$ and $\eta = \pi$

$$(17) \qquad \psi(\eta) \sim (2/\pi k \sin\eta)^{1/2}(b^2-\cos^2\eta)^{-1/4} \cos\left[k \int_0^\eta (b^2-\cos^2\eta)^{1/2} d\eta - \pi/4\right] \quad .$$

A simple calculation, using (12) and (16), shows that

$$(18) \qquad \lim_{\eta \to 0} \sin\eta\, \psi(\pi-\eta) = 2\pi^{-1} \cos\left[k \int_0^\pi (b^2-\cos^2\eta)^{1/2} d\eta\right] \quad .$$

We note that for $b = b_n$, $\psi(\eta) = W_n^{(1)}(\eta)$.

In order to expand equation (17) of Section 3 asymptotically, we evaluate the expansions contained in this Appendix at the eigenvalues b_n. When (7) is substituted into (17), we find that

$$(19)\qquad W_n^{(1)}(\eta) \sim (2/\pi k \sin\eta)^{1/2}(\cosh^2 a-\cos^2\eta)^{-1/4}\cos\Big[k\int_0^{\eta}(\cosh^2 a-\cos^2\eta)^{1/2}\,d\eta$$

$$+\,k^{1/3}\tau_n(\sinh a\cosh a)^{2/3}\int_0^{\eta}(\cosh^2 a-\cos^2\eta)^{-1/2}\,d\eta-\pi/4\Big] .$$

Similarly, the substitution of (7) into (6) yields

$$(20)\qquad V_n^{(1)}(\xi) \sim e^{i\pi/12}k^{-1/6}2^{-1}\pi^{-1/2}(\cosh^2\xi-\cosh^2 a)^{-1/4}(\sinh\xi)^{-1/2}$$

$$\exp\Big[ik\int_a^{\xi}(\cosh^2\xi-\cosh^2 a)^{1/2}\,d\xi\;-\;ik^{1/3}\tau_n(\sinh a\cosh a)^{2/3}$$

$$\int_a^{\xi}(\cosh^2\xi-\cosh^2 a)^{-1/2}\,d\xi\Big] .$$

Finally, we shall obtain the expansion of

$$\Big[V_n^{(1)}(a)\,\frac{\partial}{\partial b}\,V_n(1)(a)\Big]^{-1} .$$

To do so, we note that (3) may be differentiated with respect to both ξ and b to yield the results

$$(22) \qquad V_n^{(1)\prime}(a) \sim \pi^{-1}\zeta_n^{1/4}3^{2/3}(2\tau_n)^{-1/4}k^{5/6}(\sinh a)^{-2/3}(\cosh a)^{1/12}e^{-i\pi/3} A'(q_n)\,\frac{\partial\zeta_n}{\partial\xi} \quad ,$$

$$(23) \qquad \frac{\partial}{\partial b} V_n^{(1)}(a) \sim \pi^{-1}\zeta_n^{1/4}3^{2/3}(2\tau_n)^{-1/4}k^{5/6}(\sinh a)^{-2/3}(\cosh a)^{1/12} e^{-i\pi/3}A'(q_n)\,\frac{\partial\zeta_n}{\partial b} \quad .$$

Here ζ_n is the value of ζ at $\xi = a$, $b = b_n$. Upon differentiating (4), we find

$$(24) \qquad \frac{\partial\zeta_n}{\partial\xi} \sim -\,\zeta_n^{-1/2}(2\tau_n)^{1/2}k^{-1/3}(\sinh a)^{1/3}(\cosh a)^{-1/6} \quad .$$

We also make use of the following identities

$$(25) \qquad 2^{-1/3}k^{2/3}\zeta_n = \tau_n \quad ,$$

$$(26) \qquad \frac{\partial\zeta_n}{\partial b} = \frac{\partial\zeta_n}{\partial\tau_n}\,\frac{\partial\tau_n}{\partial b} \quad .$$

From (25) and (7), we find that

$$(27) \qquad \frac{\partial \zeta_n}{\partial b} \sim 2^{1/3} (\sinh a)^{-2/3} (\cosh a)^{1/3} \quad .$$

Upon making use of (27), (22) and (23), we obtain the desired asymptotic expansion

$$(28) \qquad \left\{ V_n^{(1)\prime}(a) \frac{\partial}{\partial b} V_n^{(1)}(a) \right\}^{-1} \sim \pi^2 e^{-i\pi/3} 3^{-4/3} 2^{-1/3} (\cosh a)^{-1/3} k^{-4/3} (\sinh a)^{5/3} \left[A'(q_n)\right]^{-2} \quad .$$

Appendix II. A theorem on back scattering

Let $\vec{E}$ be the electromagnetic field scattered back from a perfectly conducting body of revolution. The incident field $\vec{E}_i e^{-ikz}$ is assumed to be a plane wave incident along the z-axis, which is the symmetry axis of the body. We shall show that $\vec{E}$ can be expressed simply in terms of u_S and u_H, the scalar fields scattered back from the same body with the soft and hard boundary conditions respectively. In these cases the incident field is the plane wave e^{-ikz}. The result which we shall deduce is the

Back scattering theorem

$$(1) \qquad \vec{E} = \frac{1}{2} (u_S - u_H) \vec{E}_i \quad .$$

Our proof of this relation is based upon the geometrical theory of diffraction. Therefore its range of validity is the same as that of this theory.

Let us first consider the field on the reflected ray. In both scalar cases and in the electromagnetic case this field is a product of four factors. They are the value of the incident field at the point of reflection, a reflection coefficient, a geometrical or amplitude factor and a phase factor. The latter two factors are the same in all three cases. The reflection coefficient is -1 in the soft case, +1 in the hard case and -1 in the electromagnetic case. The value of this last coefficient depends upon the fact that the incident field is tangential to the surface at the point of reflection because the incident ray is normal there. From the preceding statement and the relation $-1 = \frac{1}{2}\left[-1-(+1)\right]$, we see that (1) holds for the reflected field.

To prove that (1) also holds for the diffracted field, we first treat the case of a smooth body. According to reference [3], the diffracted field $\vec{E}_d$ at any point P off the axis of the body is given by

$$(2) \qquad \vec{E}_d(P) = \sum_{j=1}^{2} \left\{ (\vec{E}_i \cdot \vec{d}^{\,j}_{20})\, \vec{d}^{\,j}_2\, v_j(P) + (\vec{E}_i \cdot \vec{d}^{\,j}_{30}) \vec{d}^{\,j}_3\, u_j(P) \right\} .$$

The unit vectors $\vec{d}^{\,j}_{20}$ and $\vec{d}^{\,j}_2$ are normal to the body at the points where the j-th diffracted ray through P is initiated, and where it leaves the surface, respectively. The unit vectors $\vec{d}^{\,j}_{30}$ and $\vec{d}^{\,j}_3$ are tangent to the body and normal to the surface ray at these same points. The scalar functions v_1 and v_2 are constructed according to the geometric theory of diffraction with the constants appropriate to a hard body. while u_1 and u_2 contain those for a soft body.

As the point P recedes from the body in a direction parallel to the axis, the points at which the rays through P leaves the body tend to limiting positions. The points of initiation of these rays also tend to limiting positions. All these limiting positions lie on that cross section of the body which is normal to the axis and which has the largest radius. Let us call the plane of this section the xy plane. It is shown in Figure 3. We have chosen the y axis to be parallel to $\vec{E}_i$. The angle between the xz plane and the plane which contains P and the symmetry axis we denote by ϕ. Then the limiting positions of the vectors in (2) are

$$\vec{d}^{\,1}_{20} = -\vec{d}^{\,2}_{20} = -\vec{d}^{\,1}_{2} = \vec{d}^{\,2}_{2} = (\cos\phi,\ \sin\phi,\ 0) \tag{3}$$

$$\vec{d}^{\,1}_{30} = -\vec{d}^{\,2}_{30} = \vec{d}^{\,1}_{3} = -\vec{d}^{\,2}_{3} = (-\sin\phi,\ \cos\phi,\ 0)\ . \tag{4}$$

When (3) and (4) are inserted into (2), it becomes

$$\begin{aligned} \vec{E}_d(P) &= -(\vec{E}_i\cdot\vec{d}^{\,1}_{20})\vec{d}^{\,1}_{20}(v_1+v_2) + (\vec{E}_i\cdot\vec{d}^{\,1}_{30})\vec{d}^{\,1}_{30}\,(u_1+u_2) \\ &= \frac{E_i}{2}\Big[-(u_1+u_2+v_1+v_2)\sin 2\phi,\, u_1+u_2-v_1-v_2+(u_1+u_2+v_1+v_2)\cos 2\phi,\, 0\Big]\ . \end{aligned} \tag{5}$$

As P tends to the axis of revolution, the expression for $\vec{E}(P)$ on the right side of (5) becomes infinite. This is because the axis is a caustic of the diffracted rays. In reference [8] there is a method for modifying fields which become infinite on an axial caustic. It applies to scalar wave functions which are proportional to sin nϕ or cos nϕ. Therefore it can be applied to each component of (5), provided the y-component is separated into a term proportional to cos 2ϕ and another independent of ϕ. To apply the method a term in sin nϕ or cos nϕ is first multiplied by the factor

$$C_n = \tfrac{1}{2}(2\pi k\rho\cos\delta)^{1/2}\sec\left[k\rho\cos\delta-(n+\tfrac{1}{2})\pi/2\right]J_n(k\rho\cos\delta)\ . \tag{6}$$

In (6) ρ is the distance of P from the axis and $\pi/2 - \delta$ is the angle which the rays make with the axis. The correct finite value of the field on the axis is obtained by letting ρ tend to zero after multiplication by C_n. The result is finite because the factor $(\rho\cos\delta)^{1/2}$ in C_n cancels

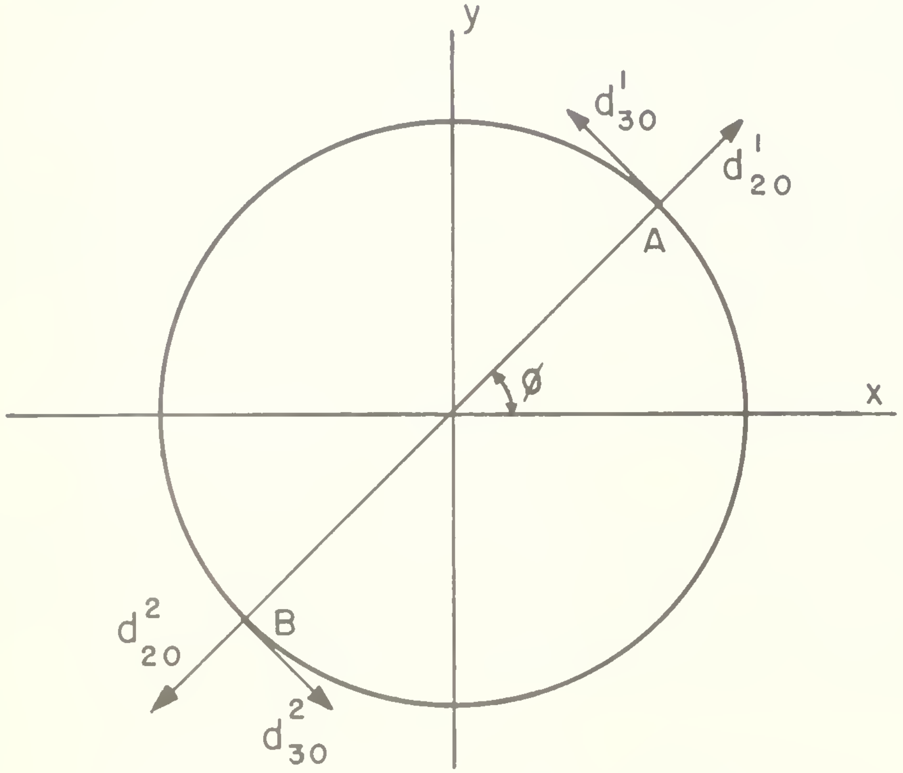

Figure 3

A cross section of the ellipsoid through its center normal to its axis of revolution. An incident ray tangent to the surface at A travels half way around the ellipsoid to B, from which point it sheds a ray which returns to the source. The vectors d^1_{20} and d^1_{30} are respectively normal and tangential to the surface at A, and both are normal to the incident ray. The vectors d^2_{20} and d^2_{30} are associated with the ray incident at B, which sheds a ray back to the source after travelling around to A. This figure is relevant when the source is far from the ellipsoid.

the singular factors in u_1, u_2, v_1 and v_2. It follows that on the axis the terms with $n \neq 0$ vanish because $J_n(0) = 0$ for $n \neq 0$. Thus only the terms with $n = 0$ in (5) remain when P is on the axis, and (5) then yields

$$(7) \qquad \vec{E}_d(P) = \frac{E_i}{2}\left[0,\ \lim_{\rho \to 0} C_o(u_1+u_2-v_1-v_2),\ 0\right] = \frac{E_i}{2}\left[0,\ u_S^d - u_H^d,\ 0\right]$$

$$= \frac{1}{2}(u_S^d - u_H^d)\ \vec{E}_i \quad .$$

The final form of (7) is the desired result (1) for the diffracted field. In obtaining it we made use of the fact that u_S^d and u_H^d, the back scattered diffracted fields for soft and hard bodies, are given by

$$(8) \qquad u_S^d = \lim_{\rho \to 0} \ C_o(u_1+u_2)$$

$$(9) \qquad u_H^d = \lim_{\rho \to 0} \ C_o(v_1+v_2)$$

The same result (7) holds when the body has an edge. The proof is similar to that given above so we shall just sketch it. For simplicity we shall confine ourselves to the case of a circular disk. If Figure 3 now represents the disk, then the field at P is again given by (5). Of course the functions u_1, u_2, v_1 and v_2 are now those appropriate to scalar diffraction by a disk. The negative sign in the first term on the right in (5) arises because the back scattered ray travels in the direction opposite that of the incident ray. Since v_1 and v_2 describe the diffraction of the tangential component of magnetic field $\vec{H}$, the opposite relations between $\vec{E}$ and $\vec{H}$ on the two rays introduce the negative sign. From (5) we obtain (7) as before, and thus (1) is proved.

We may also prove (7) for a body with an edge like that of a wedge, at which the two tangent planes meet at a non-zero angle. For that purpose the wedge diffraction coefficients given in equation (A10) of reference [8] must be used. The right side of that equation is missing a factor $-\frac{2}{n}\sin\frac{\pi}{n}$ due to a typographical error. Also in reference [8], the second columns of the diffraction matrices in equations (A12) and (A13) must be multiplied by -1.

Appendix III

Errors in a previous analysis

In reference [5] R. K. Ritt determined the asymptotic expansion of the field on the surface of a hard spheroid. He concluded that the attenuation of the diffracted wave is proportional to $R^{-2/3}\tau$. Here τ is arclength along the spheroid from the shadow boundary and R is the radius of curvature at the tip. We have seen, however, that the attenuation is actually proportional to the integral of α_m. Therefore it is proportional to the integral of $b^{-2/3}$ from the shadow boundary to the point of observation along the diffracted ray. Here b is the local radius of curvature of the generating ellipse. We have traced the reason for Ritt's erroneous conclusion to errors in the asymptotic expansions which he employs for the spheroidal functions. These expansions are derived in the appendix to reference [5] by N. D. Kazarinoff.

In that appendix (A.25) is obtained from (A.24) by setting $\lambda = -\delta^2$ and expressing γ in terms of ρ by means of (A.17). It is asserted that the function denoted by $B(x,\rho)$ in (A.25) is a bounded function of ρ. But we find from (A.17) that $\gamma = \rho(x_0^2 - 1)^{-1/2} + O(\rho^{1/3})$. From this it follows that the function denoted by $B(x,\rho)$ is not bounded. Consequently the subsequent analysis of that appendix and the resulting asymptotic expansions are not correct. However, if the ρ dependence of $B(x,\rho)$ were taken into account and the subsequent analysis modified accordingly, results in agreement with ours would be obtained.

When the ellipsoid is specialized to be a sphere there is no difference between $R^{-2/3}\tau$ and $\int_0^{\tau} b^{-2/3} d\tau$ since then $b = R$. Therefore it might be expected that the solution given by Ritt would reduce correctly to that for a sphere. Our solution does reduce to that given by W. Franz for the sphere if we let $h \rightarrow 0$ and $a \rightarrow \infty$ in such a way that $e^a h \rightarrow$ radius of the sphere. We must specialize Franz's solution to plane wave incidence and an observation point on the surface. However, Ritt's result does not reduce to that for the sphere as we see by examining a typical term in his result, given by (8.27). The reason for this discrepancy we have traced to the incorrect evaluation of $\partial y'/\partial\lambda$ carried out on page 39 of reference [5]. We see that $\partial y'/\partial\lambda$ does not have the form indicated in (A20) since the bracketed term in this equation vanishes. This follows from the fact that the vanishing of the bracketed term defines the eigenvalue. The correct evaluation is obtained by differentiating the bracketed term, which yields a result analogous to ours.

References

[1] Keller, J. B. - A geometrical theory of diffraction, Calculus of Variations and Its applications; Proc. of Symposia in Applied Math., Vol. VIII, pp. 27-52, McGraw-Hill, N.Y., 1958.

[2] Keller, J. B. - Diffraction by a convex cylinder; I.R.E. Trans. on Ant. and Prop., Symp. on Electromagnetic Wave Theory, Vol. AP-4, pp. 312-321, July 1956.

[3] Levy, B.R., and Keller, J.B. - Diffraction by a smooth object; Comm. Pure and Appl. Math., 12, No. 1, 159 (1959).

[4] Levy, B. R. - Diffraction by an elliptic cylinder; New York Univ., Inst. of Math. Sci, Div. of EM Res., Res. Report No. EM-121 (1958).

[5] Ritt, R.K. - Studies in radar cross sections XXX, The theory of scalar diffraction with application to the prolate spheroid; Univ. of Michigan Res. Inst., Scientific Report No. 4, Aug. 1958.

[6] Keller, J. B., Lewis, R. M., and Seckler, B. D. - Asymptotic solution of some diffraction problems; Comm. Pure and Appl. Math., 9, 207 (1956).

[7] Olver, F. W. J. - The asymptotic solution of linear differential equations of the second order for large values of a parameter; Phil. Trans. Royal Soc. of London, Series A, 247, 307 (1954).

[8] Keller, J. B. - Diffraction by an aperture; J. Appl. Phys., 28, No. 4, pp. 426-444, April 1957.

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